VOLUME 79 SEPARATE No. 251

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

AUGUST, 1953

ANALYSIS OF CONTINUOUS COMPOSITE STEEL AND CONCRETE BEAMS

by John Sherman

STRUCTURAL DIVISION

{Discussion open until December 1, 1953}

Copyright 1953 by the AMERICAN SOCIETY OF CIVIL ENGINEERS
Printed in the United States of America

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

THIS PAPER

--represents an effort of the Society to deliver technical data direct from the author to the reader with the greatest possible speed. To this end, it has had none of the usual editing required in more formal publication procedures.

Readers are invited to submit discussion applying to current papers. For this paper the final closing dead line appears on the front cover.

Those who are planning papers or discussions for "Proceedings" will expedite Division and Committee action measurably by first studying the printed directions for the preparation of ASCE technical papers. For free copies of these directions—describing style, content, and format—address the Manager, Technical Publications, ASCE.

Reprints from this publication may be made on condition that the full title of paper, name of author, page reference, and date of publication by the Society are given.

The Society is not responsible for any statement made or opinion expressed in its publications.

This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

ANALYSIS OF CONTINUOUS COMPOSITE STEEL AND CONCRETE BEAMS

John Sherman, M. Am. Soc. C.E.[1]

SYNOPSIS

The composite steel and concrete beam is well established in highway bridge construction. In the analysis of such bridges it is assumed that a portion of the roadway slab between adjacent stringers acts with the steel beam in resisting the compressive stresses which the positive bending moment produces in the top flange of the beam. In the case of a simply supported composite steel and concrete beam the top flange is in compression throughout the length of the beam, and if more concrete acts with the beam than was assumed the only effect is added rigidity.

In the analysis of continuous composite steel and concrete beams it is customary to assume that the steel beam takes all of the stresses due to its own weight and the weight of the roadway slab, and all of the stresses due to the negative bending moment produced by the dead and live load. The composite beam is supposed to take all of the stresses due to the positive bending moment produced by the live load, live load impact and the superimposed dead load such as wearing surface, safety curbs and handrail. In computing the stresses due to the weight of the steel beam and the weight of the roadway slab the variation in moment of inertia of the steel beam is taken into account, and in computing the stresses due to the loads taken by the composite beam the moment of inertia of the composite beam is taken to be uniform. The above manner of analysis of continuous composite beams will be called the approximate method in this paper.

It is the object of this paper to present an analysis of continuous composite steel and concrete beams based on the assumption that since the concrete does not take tension the moment of inertia of the composite beam is effective only for the portions of the span where the bending moment is positive. The moment of inertia naturally does have the abrupt change which the above assumption indicates but changes gradually the effect of which is somewhat counteracted by the fact that moment of inertia of the composite beam has a slight variation being larger at the point of maximum moment than at the points of contraflexure.

The analysis will be restricted to symmetrical two and three span continuous composite beams and from the results of the analysis it will be shown that the continuous composite beam is not as stiff as is indicated by the approximate method and that the assumptions used for the design of simply supported composite beams are not applicable to the design of continuous composite beams.

1. Engineer, American Bridge Div., United States Steel Corp., Roanoke, Va.

NOTATION

Composite beam = A steel and concrete composite beam.
E_s = The modulus of elasticity of steel.
E_c = The modulus of elasticity of concrete in compression.
$\underline{n}$ = E_s/E_c.
I_s = The moment of inertia of the steel beam.
I_c = The moment of inertia of the composite beam.
k = I_c/I_s.
M = The negative support moment in inch pounds.
M' = The maximum positive moment in inch pounds.
L = The length of the beam between supports in inches.
b = The distance center to center of the stringers in inches.
b' = The width in inches of the portion of the roadway slab which acts with the composite beam.
t = The least thickness of the roadway slab in inches.
g = The distance in inches of the point of contraflexure to the nearest support.
n = The ratio of the length of the interior span to the length of the exterior spans.

The above notation will be used throughout the text, any other notation being defined when it first occurs.

INTRODUCTION

The composite beam is composed of three essential parts: the steel beam, the concrete slab and the shear connectors.

The steel beam may be a rolled beam with or without cover plates, a plate girder or a weldment composed of three plates.

The effective width b' of the concrete portion of the composite beam is limited by the American Association of State Highway Officials Standard Specification for Highway Bridges to not exceed L/4, b or 12t. The same specification states that for concrete having a strength of 3,000 to 3,900 pounds per square inch that $\underline{n}$ is to be taken as 10 for strength calculations and 8 for figuring deflections. These values of $\underline{n}$ are for reinforced concrete beams, and although the specification does not give any value of $\underline{n}$ to be used for composite beams it is the general practice to assume that they are to be used for this type of construction. Having selected a value of $\underline{n}$ the area of the concrete portion of the composite beam in terms of steel is $b' t/\underline{n}$, and the properties of the composite beams are calculated the same as for T beams.

Since the bond between the top flange of the steel beam and the concrete slab is not dependable, mechanical shear connectors are provided to develop the horizontal shear which operates at the faying surface of the steel beam and the concrete slab. There are various types of shear connectors in use, bent or looped bars, beam or channel lugs and serpentine or spiral rods which are welded to the top flange of the steel beam.

In the analysis which follows it will be required to solve cubic and quartic equations. The roots of these equations will be M in some cases and g

in other cases, and since they will be taken as factors of the product of the load and L they will be numbers less than unity. The most direct method of solving such equations is Horner's method. This method was developed by the English mathematician W. G. Horner (1819) and is said to have been known to the Chinese in the 13th century. The method consists of forming an equation for each figure of the root, the formation of the equations being done by synthetic division. This method was selected because it requires no more work to solve equations of the type occurring in this paper than to reduce a cubic equation to the form required for a solution by radicals or to obtain the cubic resolvent of a quartic equation. Since the root we seek is always a small positive number we do not have to enter into the complexities of the general case of the method, the following being a sufficient description for the purposes of this paper.

Given the equation

$$x^3 + Qx^2 + Rx - U = 0. \qquad \text{(a)}$$

When the positive root we seek is a small decimal the first figure of the root is equal to the absolute term divided by the penultimate coefficient. Denoting the first figure of the root by p then

$$p = U/R \ . \qquad \text{(b)}$$

If we reduce all of the roots of equation (a) by p we obtain

$$x^3 + Q'x^2 + R'x - U' = 0, \qquad \text{(c)}$$

from which the second figure of the root is equal to U'/R'. This process can be continued until the degree of preciseness required is obtained.

Equation (a) is formed as follows:

1. We divide equation (a) by (x - p) and obtain for the quotient a quadratic equation which we denote by X and a remainder which is equal to U' in equation (c).

2. We divide equation X by (x - p) and obtain for the quotient an equation of the first degree which we denote by Y and a remainder which is equal to R' in equation (c).

3. We divide equation Y by (x - p) and obtain a remainder which is equal to Q' in equation (c).

In the above process if we take p equal to $\frac{Q}{3}$ then Q' will be zero and equation (c) will be in the form required for solution by radicals.

The above process will be illustrated by the following example.

Given the equation $x^3 + 4.89x^2 + 5.45x - .66 = 0$, then

$$p = .66/5.45 = .1$$

First operation.

$$\begin{array}{l|l} x - .1 & x^3 + 4.890x^2 + 5.450x - .6600 \\ & \underline{x^3 - .100x^2} \\ & \quad + 4.990x^2 + 5.450x \\ & \quad \underline{+ 4.990x^2 - .499x} \\ & \qquad + 5.949x - .6600 \\ & \qquad \underline{+ 5.949x - .5949} \\ & \qquad\qquad - .0651 = U'. \end{array}$$

$$\underline{\quad | \; x^2 + 4.99x + 5.949}$$

Second operation.

$$\begin{array}{l|l|l} \underline{x - .1} & x^2 + 4.99x + 5.949 & \underline{x + 5.09} \\ & \underline{x^2 - .10x} & \\ & \quad + 5.09x + 5.949 & \\ & \quad \underline{+ 5.09x - .509} & \\ & \qquad + 6.458 = R'. & \end{array}$$

Third operation.

$$\begin{array}{l|l} \underline{x - .1} & x + 5.09 \\ & \underline{x - .10} \\ & + 5.19 = Q'. \end{array}$$

The new equation is

$$x^3 + 5.19x^2 + 6.458x - .0651 = 0.$$

The second figure of the root is .0651/6.458 = .01 and the value of the root to two decimal places is .11.

The above procedure is simplified and compacted by the use of synthetic division as follows: writing only the literal parts of the first equation

$$\begin{array}{l|l} \underline{1 - .1} & 1 + 4.89 + 5.45 - .66 \\ & 1 + 4.99 + 5.949 - .0651. \end{array}$$

This corresponds to the first operation, the second line being formed as follows:

$$4.89 + .10 = 4.99, \quad 5.45 + .10 \times 4.99 = 5.949$$

and $$-.66 + .10 \times 5.949 = -.0651.$$

The second operation is

$$\begin{array}{l|l} \underline{1 - .1} & 1 + 4.99 + 5.949 \\ & 1 + 5.09 + 6.458 \end{array}$$

where $$4.99 + .10 = 5.09$$

and $$5.949 + .10 \times 5.09 = 6.458.$$

The third operation is

$$\begin{array}{r|l} 1 - .1 & 1 + 5.09 \\ & 1 + 5.19 \end{array}$$

where $5.09 + .10 = 5.19.$

The three steps can be combined into the following compact form and the formation of new equations continued as shown below.

$$\begin{array}{r|l} 1 - .1 & 1 + 4.89 + 5.45 \quad - .66 \\ & \underline{1 + 4.99 + 5.949 \quad - .0651} \\ & + 5.09 + 6.458 \quad\quad 0^{*} \end{array}$$

$$p' = \frac{.0651}{6.458} = .01$$

$$\begin{array}{r|l} 1 - .01 & + 5.19 + 6.510^{*} \\ & \underline{+ 5.20^{*}} \end{array}$$

$$p'' = 0.$$

The literal terms of the third equation obtained by reducing the roots of the second equation by .01 are marked by an asterisk, the absolute term of this equation is zero, therefore .11 is an exact root of the first equation.

To obtain the other two roots we reduce the roots of the first equation by .11,

$$\begin{array}{r|l} 1 - .11 & 1 + 4.89 + 5.45 \quad - .66 \\ & 1 + 5.00 + 6.00 \quad\quad 0 \ , \end{array}$$

the other two roots being contained in the quadratic equation

$$x^2 + 5x + 6 = 0,$$

and are -2 and -3.

The above method will be used throughout the text.

In comparing the results obtained by the method used in this paper with the results obtained by the approximate method the maximum unit stresses in the steel beam will be used as the basis of comparison, and since detail considerations are beyond the scope of this paper a composite beam will be used, as an example, which has the following characteristics:

1. The moment due to the uniform live load is 85% larger than the moment due to the live concentrated load.

2. The impact factor is 30%.

3. The section modulus of the composite beam for the stress in the bottom flange of the steel beam is 50% larger than the section modulus of the steel beam.

Denoting the intensity of the dead and live uniform loading by w_D and w_L, respectively, then the maximum positive moment due to the uniform dead is a factor of $w_D L^2$ which we denote by M'_D and the maximum positive moment

due to the uniform live load is a factor of $w_L L^2$ which we denote by M'_L. Denoting the relative unit stress due to the dead and live load positive moments by f' then when the uniform dead load is 50% larger than the uniform live load

$$f' = M'_D + M'_L \frac{(1.85)(1.30)}{1.5 \times 1.5} = M'_D + 1.07M'_L. \qquad (A)$$

When the uniform dead load is three times the uniform live load

$$F' = M'_D + .535M'_L. \qquad (A')$$

Denoting the maximum negative moments as M_D and M_L to be factors of $w_D L^2$ and $w_L L^2$, respectively, and the relative unit stress due the dead and live load negative moments by f then when the uniform dead load is 50% larger than the uniform live load

$$f = M_D + 1.61M_L. \qquad (B)$$

When the uniform dead load is three times the uniform live load

$$f = M_D + .815M_L. \qquad (B')$$

Equations (A), (A'), (B) and (B') will be used to calculate the relative unit stresses used for comparison with the relative unit stresses obtained by the approximate method.

THE TWO SPAN CONTINUOUS COMPOSITE BEAM

The two span beam bridge is an infrequent type of construction compared to the three span beam bridge, but since the equations for the two span beam are used in forming the equations for the case of loading in the exterior span of a three span beam a brief discussion will be given for the two span beam.

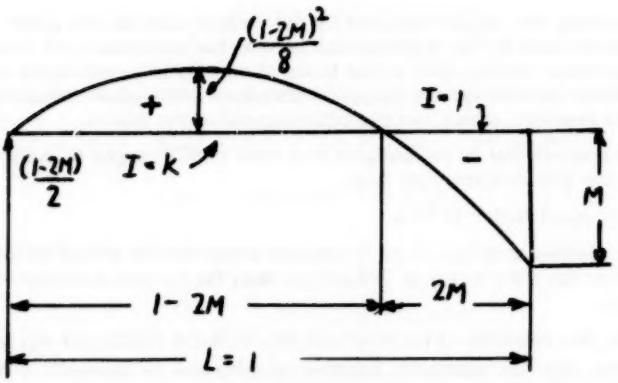

Figure 1.

In Figure 1 is shown the moment diagram due to a continuous uniformly distributed load in the exterior span of a continuous beam.

Taking the length of the span, the intensity w of the uniform load and the moment of inertia of the steel beam all as unity then the support moment M will be a factor of wL^2 and the moment of inertia of the composite beam will be k. Writing the equation for the slope P of the elastic curve at the interior support we obtain the following quartic equation

$$P = \frac{(1 - 2M)^4}{24k} - \frac{M^2}{3}(3 + 2M) + \frac{2M^3}{3}(1 - M). \qquad (1a)$$

Expanding equation (1a), collecting the terms and reversing the signs we obtain

$$M^4(k - 1) - 2M^3(k - 1) + 1.5M^2(k - 1) + .5M - .0625 + 1.5Pk = 0. \qquad (1b)$$

When P = O equation (1b) is the equation for the support moment of a symmetrical two span continuous composite beam with both spans loaded with a continuous uniformly distributed load. Substituting P = O and k = 2 in equation (1b) we obtain

$$M^4 - 2M^3 + 1.5M^2 + .5M - .0625 = 0.$$

The factor of M when k is equal to one is .125, therefore we reduce the roots of the above equation by .1 as follows

1 - .1	1 - 2	+ 1.5	+ .5	- .0625
	1 -1.90	+ 1.31	+ .631	+ .0006
	-1.80	+ 1.13	+ .744.	

Then M = .100 - .0006/.744 = .099wL^2 which is 25% less than the value of M obtained by the approximate method.

When only one span is loaded P = M/3 and the penultimate coefficient in the above equation is increased by one and we have

$$M = .10 - .1006/1.744 = .040wL^2.$$

The above value of M is less than one half the value of M when both spans are loaded which indicates that the principal of superposition fails when the loads are not in the same span.

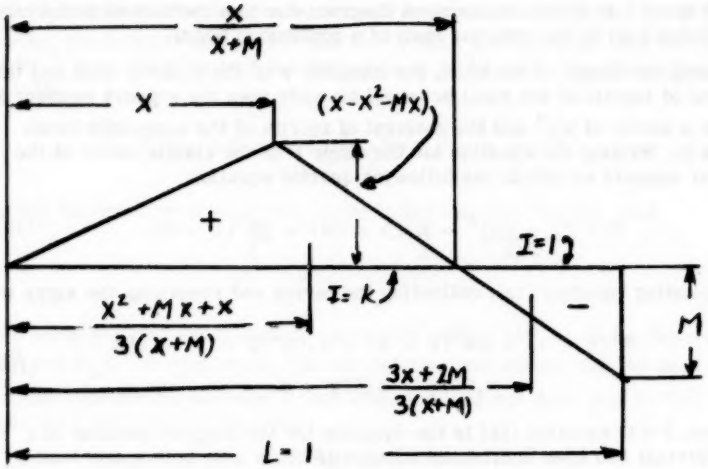

Figure 2.

In Figure 2 is shown the moment diagram due to a moving concentrated load in the exterior span of a continuous beam. In this case M will be a factor of W, where W is the value of the load.

The equation for the slope P at the interior support is

$$P = \frac{x^2(1 - x - M)}{6k(x + M)^2}(x^2 + Mx + x) - \frac{M^2(3x + 2M)}{6\ (x + M)^2}. \qquad (2a)$$

Expanding equation (a), collecting the terms and reversing the signs we obtain the following cubic equation

$$2M^3k + M^2(x^3 + 3kx + 6Pk) + 2M(x^4 + 6Pkx) - x^3 + x^5 + 6Pkx = 0. \qquad (2b)$$

When P = M/3 in equation (2b) this equation becomes the equation for the support moment of a symmetrical two span continuous composite beam with a moving load in one span, and we have

$$2M^3k + M^2(7k + x^2) + 2Mx^2(k + x^2) - x^3 + x^5 = 0. \qquad (2c)$$

For the case of a single load in each span, symmetrically placed, P = O and equation (2b) becomes

$$2M^3k + M^2x(3k + x^2) + 2Mx^4 - x^3 + x^5 = 0. \qquad (2d)$$

Taking x = .5 and k = 2 in equation (2d), dividing this equation by 4 and reducing the roots by .15 as follows

$$1 - .15 \mid 1 + .783 + .0313 - .0243$$
$$+ .933 + .1713 + .0014$$
$$+ 1.083 + .3334$$

then M = .150 - .0014/.3334 = .146 WL.

Solving equation (2c) for M we obtain M = .060WL which is less than one half the value of M for the two loads.

Table III in the appendix contains values of M for a load applied at the tenth points of one span for k = 1, 1.5, 2 and 3. In this table is given the area under the influence lines for the moving load and the corresponding values of M due to a continuous uniformly distributed load in one span calculated by equation (1b); the corresponding values of M and the areas are in close agreement from which we conclude that we can apply the principle of superposition when the loads are in the same span. In this table is given the values of the support moment M due to a continuous uniformly distributed load in both spans, and the values of the maximum positive moment M' due to uniform loading in one span.

Taking M and M' from Table III and calculating the unit stresses f and f' by equations A, A', B and B' and comparing these unit stresses with those obtained by the approximate method we obtain the errors in the approximate method given in Table I.

TABLE I

Values of k		1.5	2	3
W_D/W_L				
f	1.5	+7.8%	+12.5%	+18.7%
	3.0	+5.3%	+9.3%	+14.2%
f'	1.5	-4.7%	-7.0%	-10.5%
	3.0	-3.8%	-5.6%	-7.8%

It is seen that the unit stresses obtained by the approximate method due to the maximum negative and positive moments are larger and smaller, respectively, than obtained by the writer's method. The above errors will be increased by the superimposed dead load particularly for the outside stringers, and if more of the slab acts with the composite beam than was assumed the errors will be further increased. It therefore appears that when k is larger than 2 that some limitation should be placed on the unit stresses if the design is based on the approximate method.

THE THREE SPAN CONTINUOUS COMPOSITE BEAM

We first determine the relation between the support moments when there is no load in the interior span. In Figure 3 is shown a moment M applied at the left interior support and producing a moment CM at the right interior support.

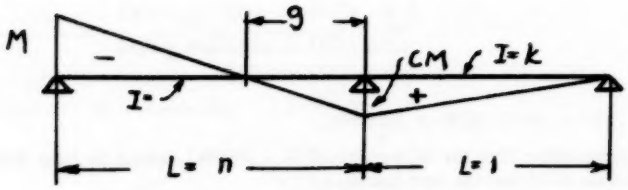

Figure 3

Equating the slopes at the right interior support we obtain the following equation

$$k(1 - g)^2 n - Cg(3n - g) = 2C. \qquad (a)$$

Noting that $g = \frac{C}{1+C}$ equation (a) becomes

$$C^3(2n + 2) + C^2(3n + 4) + 2C - nk = 0. \qquad (3)$$

Having obtained C from equation (3) the value of P with its sign reversed is

$$P = \frac{M}{6}(2 - C)n. \qquad (4)$$

Taking n = 1 and k = 1.5 in equation (3) we have

$$4C^3 + 7C^2 + 2C - 1.5 = 0.$$

Dividing the above equation by 4 and reducing its roots by .3 we have

$$\begin{array}{r|l} 1 - .3 & 1 + 1.75 + .500 - .3750 \\ & \underline{1 + 2.05 + 1.115 - .0405} \\ & + 2.35 + 1.820 \end{array}$$

Then $C = .300 + \frac{.0405}{1.82} = .321.$

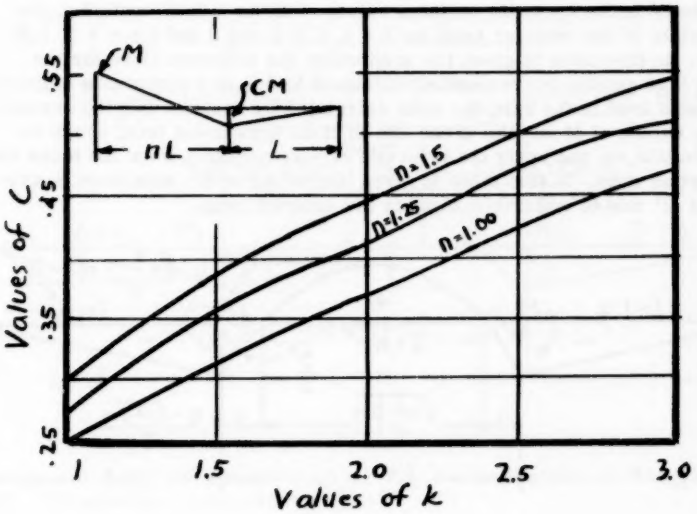

Figure 4.

Figure 4 shows the variation of C in respect to k for several values of n.

For the case of a uniform load in the exterior span of a symmetrical three span continuous composite beam we substitute equation (4) in equation (1b) and obtain

$$M^4(k-1) - 2M^3(k-1) + 1.5M^2(k-1) + .25M\{2 + (2-C)kn\} - .0625 = 0. \tag{5}$$

Taking n = 1 and k = 1.5 in equation (5), dividing by .5 and reducing the roots by .05 we have

$$\underline{1 - .05} \;\Big|\; 1 - 2.00 + 1.500 + 2.259 - .1250$$
$$\underline{\qquad - 1.95 + 1.598 + 2.338 - .0081}$$
$$- 1.90 + 1.693 + 2.422.$$

Then $M = .0500 + .0081/2.422 = .0533wL^2$.

For the case of a moving load in the exterior span of a symmetrical three span composite beam we substitute equation (4) in equation (2b) and obtain

$$M^3k\{2+(2-C)n\} + M^2x\{x^2+3k+2k(2-C)n\} + Mx^2\{2x^2+k(2-C)n\} - x^3 + x^5 = 0. \tag{6}$$

Table IV in the appendix contains values of M for a load applied at the tenth points of the exterior span for k = 1, 1.5, 2 and 3 and for n = 1, 1.25 and 1.5. In this table is given the area under the influence lines for the moving load and the corresponding values of M due to a continuous uniformly distributed load in the exterior span calculated by equation (5); the corresponding values of M and the areas are in close agreement from which we conclude that we can apply the principle of superposition when the loads are in the same span. In this table is given the values of the maximum positive moment M' due to uniform loading in the exterior span.

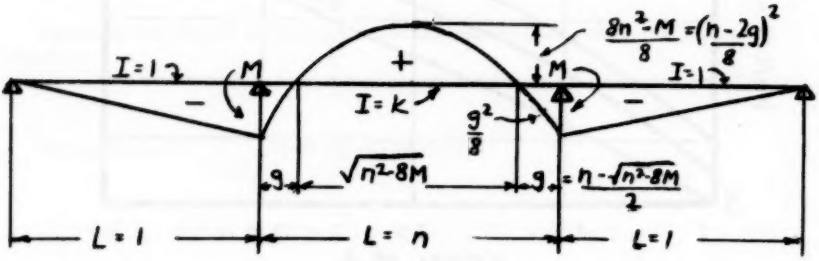

Figure 5.

Figure 5 shows the moment diagram due to a continuous uniformly distributed load in the interior span.

The value of g is

$$g = n - \frac{(n^2 - 8M)^{\frac{1}{2}}}{2} \tag{7}$$

from which

$$M = g\left(\frac{n - g}{2}\right). \tag{8}$$

The equation for the condition that the slope of the elastic curve is the same for each interior support is

$$\frac{M}{3} + \frac{Mg}{2} - \frac{g^3}{12} - \frac{(n - 2g)^3}{24k} = 0. \tag{9}$$

Substituting equation (8) in equation (9) we have

$$8g^3(k - 1) - 2g^2\{6n - k(3n - 2)\} + 2gn(3n + 2k) - n^3 = 0. \tag{10}$$

Solving equation (10) for g and substituting g in equation (8) we obtain M.

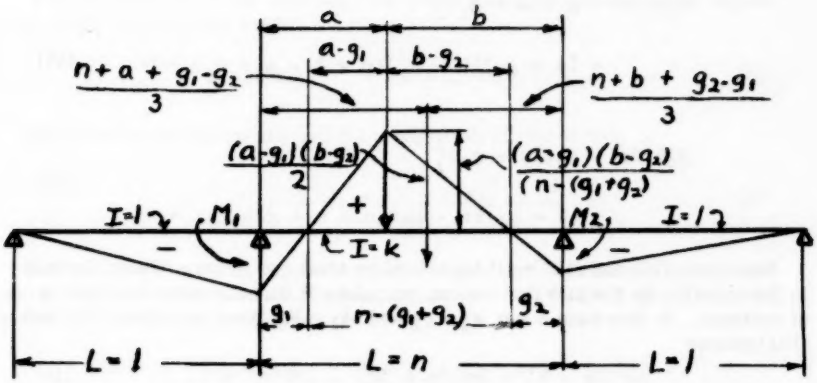

Figure 6.

Figure 6 shows the moment diagram due to a moving load in the interior span. The equations for g_1 and g_2 are

$$g_1 = \frac{M_1 n}{b - M_1 + M_2} \qquad g_2 = \frac{M_2 n}{a + M_1 - M_2} \tag{11}$$

Equating the slopes at the interior supports we obtain the following two equations

$$\frac{M_1}{3} + \frac{M_1 g_1 (3n - g_1)}{6n} - \frac{(a - g_1)(b - g_2)(n + b - g_1 + g_2)}{6nk} + \frac{M_2 g_2^2}{6n} = 0 \tag{12}$$

$$\frac{M_2}{3} + \frac{M_2 g_2 (3n - g_2)}{6n} - \frac{(a - g_1)(b - g_2)(n + a + g_1 - g_2)}{6nk} + \frac{M_1 g_1^2}{6n} = 0. \tag{13}$$

The penultimate term in each of the above equations is small in comparison with the other terms; we therefore discard these terms and write

$$M_1 k\left\{2n + g_1(3n - g_1)\right\} - (a - g_1)(b - g_2)(n + b - g_1 + g_2) = 0 \qquad (14)$$

$$M_2 k\left\{2n + g_2(3n - g_2)\right\} - (a - g_1)(b - g_2)(n + a + g_1 - g_2) = 0. \qquad (15)$$

Equations (14) and (15) must be solved by trial and error. We are helped in the solution by the fact that we can calculate M directly when the load is at midspan. In this case $a = b$, $g_1 = g_2$ and $M_1 = M_2$, then equations (12) and (13) become

$$6M^2(k - 1) + Mn(2n + 3k) - .375n^3 = 0. \qquad (16)$$

Calculating M for $x = .5n$ by equation (16) and drawing the influence line for $k = 1$, we use this influence line to approximate M_1 and M_2 for any other value of x. Using these approximate values we calculate g_1 and g_2 by equations (11) and use these values of g to obtain new values of M_1 and M_2 from equations (14) and (15); from the new values of M we recalculate g and repeat the procedure until equations (11), (14) and (15) are in close agreement.

Table V in the appendix contains values of M for a load applied at the tenth points of the interior span for $k = 1$, 1.5, 2 and 3 and $n = 1$, 1.25 and 1.5. In this table is given the area under the influence line for the moving load and the corresponding values of M due to a continuous uniformly distributed load in the interior span calculated by equations (8) and (10); the corresponding values of M and the areas are in close agreement from which we can conclude that we can apply the principle of superposition when the loads are in the same span. Also given in this table is the value of M' the maximum positive moment due to uniform loading in the interior span.

For the case of a continuous uniformly distributed load in all three spans we refer to Figure 5 and write for the slope at the left interior support

$$P = -M\left\{\frac{n - (n^2 - 8M)^{\frac{1}{2}}}{4}\right\} + \frac{1}{96}\left\{n - (n^2 - 8M)\right\}^3 - \frac{(n^2 - 8M)^{3/2}}{24k}. \qquad (a)$$

The above equation when substituted in equation (1b) will result in a complex equation. To avoid this we rationalize equation (a) by means of the binominal expansion of the radicals and obtain an approximate solution which investigations have shown to be sufficiently precise for our purposes.

Since the first term is small compared to the third term we take the first two terms of the binominal expansion of the radical in the first term and write

$$(n^2 - 8M)^{\frac{1}{2}} = n - \frac{4M}{n}. \qquad (b)$$

For the third term we take the first three terms of the binominal expansion of the radical and write

$$(n^2 - 8M)^{3/2} = n^3 - 12Mn + 24\frac{M^2}{n}. \qquad (c)$$

Substituting equations (b) and (c) in equation (a) we obtain

$$P = -\frac{24M^2(k-1) - 12Mn^2 + n^4}{6kn}. \qquad (d)$$

Substituting P with its sign reversed in equation (1b) and discarding all powers of M above the second power, because the coefficient of M^2 is large, we obtain the following equation for M

$$24M^2(k-1)(n+1) + 4Mn(3n+2) - n(n^3+1) = 0. \qquad (17)$$

For the case of uniform moment of inertia equation (17) becomes

$$M = \frac{(n^3+1)wL^2}{4(3n+2)}. \qquad (18)$$

Table VI in the appendix contains values of M calculated by equation (17) for k = 1, 1.5 and 2 and n = 1, 1.25 and 1.5, and also the values of M'_e and M'_i the maximum positive moments in the exterior and interior spans, respectively, due to a continuous uniformly distributed load in all three spans.

Taking the moments from Tables IV, V and VI and calculating the unit stresses f and f' by equations A, A', N and B' and comparing these unit stresses with those obtained by the approximate method we obtain the errors in the approximate method given in Table II.

TABLE II

Values of k			1.5		2		3	
			w_D/w_L	w_D/w_L	w_D/w_L	w_D/w_L	w_D/w_L	w_D/w_L
Moment	n		1.5	3	1.5	3	1.5	3
	1.00	f'_i	- 9.5%	- 7.7%	-17.2%	-13.9%	-23.8%	-18.5%
M'_i	1.25	f'_i	- 7.4%	- 5.5%	-12.2%	- 9.4%	-20.6%	-15.6%
	1.50	f'_i	- 8.7%	- 6.2%	-14.5%	-10.2%	-21.1%	-15.0%
	1.00	f'_e	- 3.3%	- 3.1%	- 5.5%	- 3.8%	- 8.3%	- 5.3%
M'_e	1.25	f'_e	- 4.0%	- 2.4%	- 6.4%	- 4.9%	- 8.0%	- 5.8%
	1.50	f'_e	- 3.1%	- 2.8%	- 4.4%	- 3.7%	- 8.1%	- 5.6%
	1.00	f	+11.4%	+ 8.2%	+20.6%	+13.7%	+20.0%	+20.4%
M	1.25	f	+ 9.5%	+ 7.3%	+16.4%	+11.3%	+27.6%	+19.8%
	1.50		+10.7%	+ 7.7%	+17.8%	+12.7%	+26.0%	+18.7%

It is seen that as was the case with the two span continuous composite beam that the approximate method indicates stresses due to the positive moment as too small and the stresses due to the negative moment as too large when compared to the stresses determined from the moments calculated by the equations in this paper. The errors in the exterior span are small but the errors in the interior span indicate that some limitation should

be placed on the unit stresses used for design when k is larger than 1.5.

In a three span composite beam the maximum live load deflection occurs at the center of the interior span when the uniform live load covers the interior span. The equation for this deflection by the approximate method is

$$y = \frac{5wn^4L^4}{384E_sI_c} - \frac{Mn^2L^2}{8E_sI_c} \qquad (19)$$

Referring to Figure 5 and assuming that the moment of inertia of the composite beam operates only in the positive portions of the moment diagram then

$$y = \frac{wL^4(n - 2g)^3(5n + 6g)}{384E_skI_s} - \frac{Mg^2L^2}{6E_sI_s} + \frac{wg^4L^4}{24E_sI_s} . \qquad (20)$$

Taking $n = 1.5$ and $k = 3$ then $M = .130wL^2$ for the condition of uniform moment of inertia, substituting these values in equation (19) we have

$$y = \frac{wL^4}{3E_sI_s}(.01302x1.5^2 - .125x.130) = \frac{.0093wL^4}{E_sI_s} .$$

To use equation (20) we find that $M = .0753wL^2$ and $g = .11L$ then

$$y = \frac{wL^4}{E_sI_s}\left(\frac{1.28^3x8.16}{384x3} - \frac{.0753x.11^2}{6} + \frac{.11^4}{24}\right)$$

$$= \frac{.0150wL^4}{E_sI_s}$$

which is 60% larger than the deflection obtained by the approximate method. When $n = 1$ and $k = 1.5$ the deflection obtained by equation (20) is 35% larger than the deflection obtained by the approximate method.

For the case of a concentrated load at the center of the interior span the deflection at the load by the approximate method is

$$y = \frac{Wn^3L^3}{48E_sI_c} - \frac{Mn^2L^2}{8E_sI_c} . \qquad (21)$$

Referring to Figure 6, taking $a = b = L/2$ and assuming that the moment of inertia of the composite beam operates only in the positive portions of the moment diagram then

$$y = \frac{WL^3(n - 2g)^2(n + g)}{E_skI_s} - \frac{Mg^2}{6E_sI_s} \qquad (22)$$

Taking a beam having the same proportions as in the preceding example the deflection using equation (21) is

$$y = \frac{.0117WL^3}{E_sI_s} .$$

The deflection using equation (22) is

$$y = \frac{.0164WL^3}{E_s I_s}$$

which is 40% larger than the deflection obtained by the approximate method.

It is seen from the foregoing examples that the composite beam is not as stiff as calculations by the approximate method indicate.

CONCLUSIONS

It was stated in the introduction that the American Association of State Highway Officials Standard Specification for Highway Bridges does not give a value of n to be used with composite beams. In the case of a simply supported composite beam the use of the values of n given in this specification results in a conservative design, it being customary to design the shear connectors for the dead and live load shear to provide for the possibility of more concrete acting with the composite beam than was assumed. In the case of a continuous composite beam it has been shown in this paper that there is a definite relation between the amount of concrete which acts with the composite beam and the moment distribution, from which it is concluded that the design assumptions used for simply supported composite beams will not result in conservative designs when applied to the design of continuous composite beams.

To obtain a conservative design for a continuous composite beam when the moment distribution is determined by the method in this paper n should be taken larger for stress calculations and for computing deflections than the value used to determine the moment distribution. When the approximate method is used to determine the moment distribution the value of n used for stress calculations and figuring deflections should be taken larger than 10 and k should be limited to three.

E_c is subject to variations and therefore I_c and k are approximate quantities, and since in the case of a continuous composite beam the moment distribution is determined by k, which in turn depends on E_c, the analysis of continuous composite beams is not on a basis as precise as is the analysis of steel continuous beams. Therefore some limitation should be placed on the design of continuous composite beams by specifications which would result in conservative designs, and the writer recommends that such specifications be prepared and adopted.

APPENDIX

TABLE III

Factors of M for a moving load in one span of a symmetrical two span continuous composite beam.

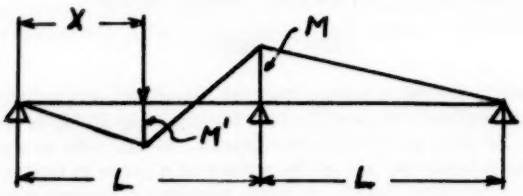

Figure 7.

x	k = 1	k = 1.5	k = 2	k = 3
.10	.025	.019	.015	.011
.20	.048	.038	.031	.026
.30	.068	.055	.041	.034
.40	.084	.066	.053	.040
.50	.094	.073	.060	.045
.60	.096	.075	.062	.046
.70	.089	.067	.057	.043
.80	.072	.055	.047	.035
.90	.043	.034	.028	.016
Area	.0625	.0487	.0397	.0297
M	.0625	.0485	.0400	.0296
$\underline{M}$	.1250	.1110	.0990	.0840
$\underline{M'}$	.0940	.1020	.1060	.1110
$\underline{M'}$	.0703	.0755	.0804	.0865

In the table above $\underline{M}$ and $\underline{M'}$ are the negative support moment and maximum positive moment, respectively, due to a continuous uniformly distributed load in both spans.

TABLE IV

Factors of M for a moving load in the exterior span of a symmetrical three span continuous composite beam.

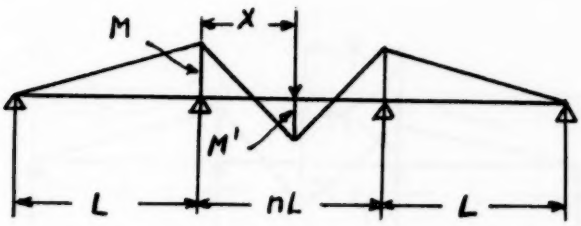

Figure 8.

	n = 1 Values of k.				n = 1.25 Values of k.				n = 1.5 Values of k.			
x	1	1.5	2	3	1	1.5	2	3	1	1.5	2	3
.10	.026	.021	.018	.014	.024	.018	.014	.013	.022	.017	.015	.012
.20	.051	.041	.035	.027	.046	.035	.032	.025	.043	.032	.026	.021
.30	.073	.058	.049	.036	.065	.049	.043	.035	.060	.046	.039	.030
.40	.089	.071	.061	.046	.083	.063	.053	.041	.075	.058	.049	.038
.50	.100	.080	.068	.052	.091	.070	.059	.045	.083	.062	.054	.043
.60	.102	.082	.071	.056	.093	.072	.061	.048	.085	.068	.056	.046
.70	.095	.077	.064	.050	.086	.068	.058	.045	.079	.062	.053	.041
.80	.076	.061	.052	.041	.069	.055	.048	.037	.064	.049	.043	.034
.90	.046	.038	.031	.025	.041	.033	.029	.022	.037	.030	.026	.020
Area	.0666	.0535	.0452	.0349	.0600	.0467	.0403	.0307	.0552	.0427	.0365	.0287
M	.0667	.0533	.0455	.0355	.0600	.0467	.0404	.0306	.0553	.0432	.0363	.0283
M'	.0938	.0996	.1034	.1078	.068	.1028	.1067	.1101	.0990	.0914	.1064	.1113

TABLE V

Factors of M for a moving load in the interior span of a symmetrical three span continuous composite beam.

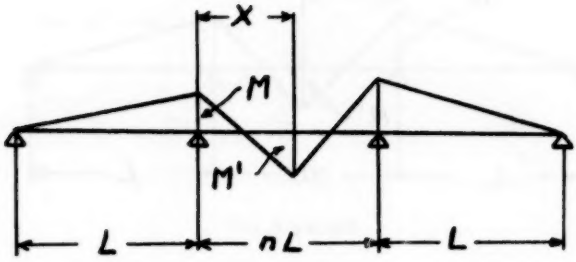

Figure 9.

	n = 1 Values of k.				n = 1.25 Values of k.				n = 1.5 Values of k.			
x	1	1.5	2	3	1	1.5	2	3	1	1.5	2	3
.10	.039	.031	.027	.021	.054	.045	.037	.027	.070	.059	.045	.039
.20	.064	.052	.043	.034	.088	.074	.067	.048	.114	.094	.075	.063
.30	.077	.062	.051	.040	.106	.085	.073	.056	.136	.113	.097	.078
.40	.080	.064	.054	.042	.109	.090	.076	.059	.138	.116	.099	.080
.50	.075	.060	.052	.038	.102	.085	.071	.057	.130	.108	.092	.074
.60	.064	.052	.046	.035	.086	.072	.058	.049	.109	.091	.081	.064
.70	.049	.041	.035	.027	.065	.057	.048	.038	.082	.070	.063	.051
.80	.032	.027	.023	.020	.042	.037	.035	.025	.051	.042	.038	.031
.90	.015	.016	.011	.009	.019	.018	.015	.012	.024	.020	.018	.017
Area	.0500	.0408	.0344	.0267	.0848	.0710	.0623	.0467	.1300	.1027	.0923	.0753
M	.0500	.0408	.0344	.0269	.0848	.0710	.0623	.0469	.1300	.1076	.0923	.0753
M'	.0750	.0840	.0920	.0980	.1110	.1240	.1330	.1480	.1510	.1740	.1890	.2060

TABLE VI

Factors of M for a continuous uniformly distributed load in all three spans.

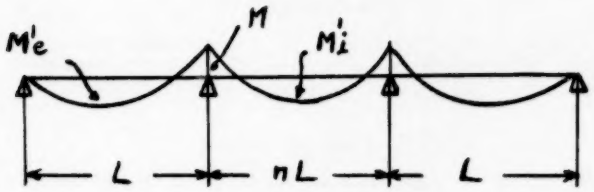

Figure 10.

	Factors of M. Values of k.				Factors of M'$_i$. Values of k.				Factors of M'$_e$. Values of k.			
n	1	1.5	2	3	1	1.5	2	3	1	1.5	2	3
1.00	.100	.090	.084	.073	.025	.035	.041	.052	.080	.084	.087	.091
1.25	.126	.118	.108	.098	.069	.077	.087	.102	.070	.073	.077	.081
1.50	.168	.151	.131	.115	.113	.130	.150	.166	.055	.061	.068	.073

AMERICAN SOCIETY OF CIVIL ENGINEERS

OFFICERS FOR 1953

PRESIDENT

WALTER LEROY HUBER

VICE-PRESIDENTS

Term expires October, 1953:
GEORGE W. BURPEE
A. M. RAWN

Term expires October, 1954:
EDMUND FRIEDMAN
G. BROOKS EARNEST

DIRECTORS

Term expires October, 1953:
KIRBY SMITH
FRANCIS S. FRIEL
WALLACE L. CHADWICK
NORMAN R. MOORE
BURTON G. DWYRE
LOUIS R. HOWSON

Term expires October, 1954:
WALTER D. BINGER
FRANK A. MARSTON
GEORGE W. McALPIN
JAMES A. HIGGS
I. C. STEELE
WARREN W. PARKS

Term expires October, 1955:
CHARLES B. MOLINEAUX
MERCEL J. SHELTON
A. A. K. BOOTH
CARL G. PAULSEN
LLOYD D. KNAPP
GLENN W. HOLCOMB
FRANCIS M. DAWSON

PAST-PRESIDENTS

Members of the Board

GAIL A. HATHAWAY
CARLTON S. PROCTOR

TREASURER

CHARLES E. TROUT

EXECUTIVE SECRETARY

WILLIAM N. CAREY

ASSISTANT TREASURER

GEORGE W. BURPEE

ASSISTANT SECRETARY

E. L. CHANDLER

PROCEEDINGS OF THE SOCIETY

HAROLD T. LARSEN
Manager of Technical Publications

DEFOREST A. MATTESON, JR.
Assoc. Editor of Technical Publications

PAUL A. PARISI
Asst. Editor of Technical Publications

COMMITTEE ON PUBLICATIONS

LOUIS R. HOWSON

FRANCIS S. FRIEL
GLENN W. HOLCOMB
I. C. STEELE
FRANK A. MARSTON
NORMAN R. MOORE

• Readers are urged to submit discussion applying to current papers. Forty free Separates per year are allotted to members. Mail the coupon order form found in the current issue of *Civil Engineering*.